Optimality Conditions for Distributive Justice

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Abstract

We analyze utilitarian and Rawlsian criteria for distribution of limited resources by deriving optimality conditions for appropriate optimization problems. We assume that some individuals are more productive than others, so that an inequitable distribution of resources creates greater overall utility. We derive conditions under which a distribution of wealth (a) maximizes utility, (b) maximizes a utility function that accounts for the social cost of inequality, and (c) satisfies a lexmax criterion that reflects the Rawlsian difference principle. We show that a utilitarian solution (a) can distribute resources equally only when all individuals have the same marginal productivity. Equality is possible under (b) in a diverse population when the cost of inequality is sufficiently large. Equality is possible under the Rawlsian option (c) when no segment of society has a much greater average productivity than the rest. Equality is more likely to be consistent with Rawlsian justice when there are rapidly decreasing returns to greater investment in productivity, when the most productive individuals are not much more productive than the average, and, ironically, when people are more interested in getting rich.

1 Introduction

Utilitarianism and the Rawlsian difference principle imply different criteria for distributive justice, but both can be viewed as mathematical optimization problems. Utilitarianism maximizes a social utility function whose arguments represent wealth distributed to individuals. The Rawlsian difference principle calls for a lexicographic maximum of the utilities allotted to individuals. This suggests that the theory of optimization might provide some insight into the conditions under which a distribution of wealth satisfies a utilitarian or a Rawlsian criterion.

In particular, we use classical optimality conditions to analyze distributions over nonidentical individuals. This is a departure from most axiomatic treatments of distributive justice, which assume that individuals are indistinguishable (Blackorby, Bossert, & Donaldson 2002). This capability allows us to study one of the perennial issues of distributive justice—the extent to which an efficient distribution of wealth requires inequality. It is sometimes argued that more utility is created when greater shares of wealth are alloted to individuals who are more talented, more productive, or work harder.

We use the modeling device of assigning to each individual i a productivity function $u_i(\alpha)$ that measures the total utility eventually created when individual i is initially aloted wealth α. We then find the distribution of initially available wealth that ultimately results in the greatest total utility. We investigate the degree of inequality that is required to maximize utility, and well as conditions under which a completely egalitarian distribution maximizes utility. We perform a similar analysis when the calculation of utility accounts for the fact that excessive inequality may disrupt social harmony and ultimately reduce total utility. In particular, we determine when the cost of inequality is high enough so that an egalitarian distribution maximizes utility.

The Rawlsian difference principle states roughly that inequality should be tolerated only when it is necessary and sufficient to result in greater utility for everyone. We follow the common practice of interpreting this as an imperative to find a lexmax distribution. To do so we suppose that individuals have a common utility function $v(\alpha)$ that measures the personal utility that results when an individual is alloted wealth α. We further suppose that the fraction of the total utility that is eventually enjoyed by an individual is proportional to the utility of that individual's initial wealth allocation. Thus we view the initial allocation of resources to individuals as assigning social status and privilege. We derive conditions under which a distribution of wealth satisfies the lexmax criterion, as well as conditions under which the lexmax distribution is completely egalitarian.

2 Utilitarian Distribution

We first formulate the utilitarian problem. Let the utility generated by person i from wealth x_i be $u_i(x_i)$. If the total resource budget is 1, the problem of distributing wealth to maximize utility is

$$\begin{array}{ll} \max \displaystyle\sum_{i=1}^n u_i(x_i) & (a) \\ \displaystyle\sum_{i=1}^n x_i = 1 & (b) \\ x_i \geq 0, \ \text{all } i & (c) \end{array} \qquad (1)$$

If we associate Lagrange multiplier λ with the constraint (1b), any optimal solution of (1) in which each $x_i > 0$ must satisfy

$$u_i'(x_i) - \lambda = 0,\ i = 1, \ldots, n$$

Eliminating λ yields

$$u_1'(x_1) = \cdots = u_n'(x_n) \tag{2}$$

Thus a wealth distribution is optimal only when the marginal productivity of wealth is the same for everyone.

Assume that individuals $1, \ldots, n$ are indexed by increasing marginal productivity:

$$u_{i+1}'(\alpha) \geq u_i'(\alpha) \text{ for all } \alpha \geq 0 \text{ and } i = 1, \ldots, n-1 \tag{3}$$

In this case, (2) is satisfied only if $x_1 \leq \cdots \leq x_n$. Thus the less productive individuals receive less wealth, as one might expect. Furthermore, a utilitarian distribution is completely egalitarian ($x_1 = \cdots = x_n = 1/n$) only when the marginal productivities are equal:

$$u_1'(1/n) = \cdots = u_n'(1/n)$$

To obtain some idea of how skewed the wealth distribution might be, it is helpful to assume a specific form

$$u_i(x_i) = c_i x_i^p \tag{4}$$

for the utility functions, where $p \geq 0$ and each $c_i \geq 0$. Here c_i indicates the productivity of person i. When $p = 1$, person i produces utility in proportion to the wealth received. When $0 < p < 1$, greater wealth has decreasing marginal utility, and $p = 0$ indicates inability to use wealth to create utility. If individuals are indexed in order of marginal productivity, we have that $c_1 \leq \cdots \leq c_n$.

Since an optimal solution of (1) in which each $x_i > 0$ must satisfy (1b) and (2), it is

$$x_i = c_i^{\frac{1}{1-p}} \left(\sum_{j=1}^{n} c_j^{\frac{1}{1-p}} \right)^{-1} \tag{5}$$

when $0 \leq p < 1$. When $p \geq 1$, it is clear on inspection that an optimal solution sets $x_n = 1$ and $x_i = 0$ for $i = 1, \ldots, n-1$.

Then the optimal distribution is completely unequal when utility generated is proportional to wealth ($p = 1$). The most productive member of society receives all the wealth. The distribution becomes increasingly egalitarian as p approaches zero, reaching in the limit a distribution in which each person i receives wealth in proportion to c_i. Thus the most egalitarian distribution that is possible in this utilitarian model is one in which people receive wealth in proportion to their productivity. Moreover, this occurs only in the limiting case when the utility generated is independent of the wealth received ($p = 0$). When $0 < p \leq 1$, a utilitarian distribution can be completely egalitarian ($x_1 = \cdots = x_n$) only when $c_1 = \cdots = c_n$. When $p > 1$, one individual must receive all the wealth even when $c_1 = \cdots = c_n$.

Using this model, more egalitarian distributions are less efficient. In an optimal distribution with $0 \leq p < 1$, the total utility is

$$\left(\sum_{i=1}^{n} c_i^{\frac{1}{1-p}} \right)^{1-p} \tag{6}$$

In a completely egalitarian distribution, each $x_j = 1/n$, and the total utility is

$$\left(\frac{1}{n} \right)^p \sum_{i=1}^{n} c_i \tag{7}$$

The ratio (7)/(6) indicates the utility cost of an egalitarian distribution.

3 Cost of Inequality

The rudimentary utilitarian model above implies that a utilitarian solution can result in considerable inequity when individuals have different abilities. A classical defense of utilitarianism, however, is that excessive inequity generates disutility by contributing to social disharmony. The model (1) does not account for any such cost of inequality. A more adequate model may result in utilitarian wealth distributions that are more equitable.

A simple way to try to capture the cost of inequity is to model it as a proportional to the total range of incomes. The model (1) becomes

$$\begin{aligned} &\max \sum_{i=1}^{n} u_i(x_i) - \beta \left(\max_i \{x_i\} - \min_i \{x_i\} \right) \\ &\sum_{i=1}^{n} x_i = 1 \\ &x_i \geq 0,\ \text{all } i \end{aligned} \tag{8}$$

Presumably, a positive cost factor β could result in utilitarian solutions that distribute wealth more equally. It is also interesting to derive how large β must be to result in a completely egalitarian distribution.

The analysis is easier if we linearize problem (8) using the following lemma. We again assume that individuals are indexed by increasing marginal productivity, as in (3).

Lemma 1 *If the utility functions u_i satisfy (3), and (8) has an optimal solution, then the following problem has the same optimal value as (8):*

$$\begin{aligned} &\max \sum_{i=1}^{n} u_i(x_i) - \beta(x_n - x_1) && (a) \\ &\sum_{i=1}^{n} x_i = 1 && (b) \\ &x_i \leq x_{i+1},\ i = 1, \ldots, n-1 && (c) \\ &x_i \geq 0,\ \textit{all } i && (d) \end{aligned} \tag{9}$$

Proof. Let x^* be an optimal solution of (8) with optimal value U^*. If $x_j^* > x_k^*$ for some j, k with $j < k$, then create a new solution x^1 defined by $x_j^1 = x_k^*$, $x_k^1 = x_j^*$, and $x_i^1 = x_i^*$ for $i \neq j, k$. If U_1 is the objective function value of solution x^1 in (8), then

$$U_1 = U^* + u_j(x_k^*) - u_j(x_j^*) + u_k(x_j^*) - u_k(x_k^*) \tag{10}$$

But due to (3),

$$u_k(x_j^*) - u_k(x_k^*) \geq u_j(x_j^*) - u_j(x_k^*)$$

because $j < k$. This and (10) imply that $U_1 \geq U^*$. Now if $x_j^1 > x_k^1$ for some j, k with $j < k$, create a new solution x^2 in the same manner, and observe again that the objective function of (8) does not decrease. Continue with the sequence $x^1, \ldots, x^t$ until $x_1^t \leq \cdots \leq x_n^t$. Then x^t is feasible in the problem

$$\begin{aligned}
&\max \sum_{i=1}^{n} u_i(x_i) - \beta \left(\max_i\{x_i\} - \min_i\{x_i\} \right) \\
&\sum_{i=1}^{n} x_i = 1 \\
&x_i \leq x_{i+1},\ i = 1, \ldots, n-1 \\
&x_i \geq 0,\ \text{all } i
\end{aligned} \tag{11}$$

and has an objective function optimal value no less than U^*. But (11) has an optimal value no greater than U^* because it is more highly constrained than (8). Thus (8) and (11) have the same optimal value. But (11) is obviously equivalent to (9), which implies that (8) and (9) have the same optimal value, as claimed.

To characterize optimal solutions of (9), we associate Lagrange multiplier λ with (9b) and multipliers $\mu_1, \ldots, \mu_{n-1}$ with the constraints in (9c). The Karush-Kuhn-Tucker (KKT) optimality conditions imply that x is optimal in (9) only if there are a value of λ and nonnegative values of $\mu_1, \ldots, \mu_{n-1}$ such that

$$\begin{aligned}
&u_1'(x_1) + \beta - \lambda - \mu_1 = 0 \\
&u_i'(x_i) - \lambda + \mu_{i-1} - \mu_i = 0,\ i = 2, \ldots, n-1 \\
&u_n'(x_n) - \beta - \lambda + \mu_{n-1} = 0
\end{aligned} \tag{12}$$

where $\mu_i = 0$ if $x_i < x_{i+1}$ in the solution.

We first examine the case in which each individual receives a different wealth allotment x_i. In this case each $\mu_i = 0$, and we can eliminate λ from (12) to obtain

$$\begin{aligned}
&u_2'(x_2) = \cdots = u_{n-1}'(x_{n-1}) \\
&u_1'(x_1) = u_2'(x_2) - \beta \\
&u_n'(x_n) = u_2'(x_2) + \beta
\end{aligned}$$

Thus all individuals who are not at the extremes of the distribution have equal marginal productivity in a utilitarian distribution, just as they do in the solution of the original model (1). The individual at the bottom of the distribution, however, has marginal productivity that is β smaller than that of those in the middle, while the individual at the top has marginal productivity that is β larger than that of those in the middle. This tends to result in somewhat larger allotment for the individual at the bottom, and a smaller allotment for the one at the top. Since the remaining individuals are forced to lie between these extremes, the net result is a distribution that is less skewed than in the original model.

4 Equality in a Utilitarian Distribution

We can also determine what value of β results in a completely egalitarian model. In this case the multipliers μ_i can be nonzero. Again eliminating λ from the KKT conditions (12), we get

$$\begin{aligned}
&2\mu_1 - \mu_2 = d_1 \\
&\mu_1 + \mu_i - \mu_{i+1} = d_i,\ i = 2, \ldots n-2 \\
&\mu_1 + \mu_{n-1} = d_{n-1}
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
&d_i = u_1'(x_1) - u_{i+1}'(x_{i+1}) + \beta,\ i = 1, \ldots, n-1 \\
&d_{n-1} = u_1'(x_1) - u_n'(x_n) + 2\beta
\end{aligned} \tag{14}$$

It can be checked that the following solves (13)

$$\mu_k = \frac{k}{n} \sum_{i=k}^{n-1} d_i - \left(1 - \frac{k}{n}\right) \sum_{i=1}^{k-1} d_i \tag{15}$$

for $k = 1, \ldots, n-1$. Substituting (14) into (15), we get

$$\mu_k = \beta + \left(1 - \frac{k}{n}\right) \sum_{i=1}^{k} u_i'(x_i) - \frac{k}{n} \sum_{i=k+1}^{m} u_i'(x_i) \tag{16}$$

for $k = 1, \ldots, n-1$.

We now consider an egalitarian solution, in which each $x_i = 1/n$. Since each $\mu_i \geq 0$ in an optimal solution, we obtain the following from (16).

Theorem 2 *Suppose that individuals are indexed in order of increasing marginal productivity. Then an utilitarian distribution in the model (8) is egalitarian ($x_1 = \cdots = x_n$) only if*

$$\beta \geq \frac{k}{n} \sum_{i=k+1}^{m} u_i'(1/n) - \left(1 - \frac{k}{n}\right) \sum_{i=1}^{k} u_i'(1/n) \tag{17}$$

for $k = 1, \ldots, n-1$.

This may be easier to interpret for the specific productivity functions defined earlier.

Corollary 3 *If the productivity function u_i are given by (4), a utilitarian distribution in the model (8) is egalitarian only if*

$$\beta \geq \frac{p}{n^p} k(n-k) \left(\frac{1}{n-k} \sum_{i=k+1}^{n} c_i - \frac{1}{k} \sum_{i=1}^{k} c_i \right)$$

for $k = 1, \ldots, n-1$.

Thus to determine the minimum β required to ensure equality, we examine each group of k smallest coefficients $c_1, \ldots, c_k$. The value of β depends on the difference between the average of these coefficients and the average of the remaining coefficients. Thus if there is a group of individuals who are much less productive on the average than the remaining individuals, relative to the overall range of productivities, a larger β is required to ensure inequality. This could occur in a two-class society with a relatively homogeneous underclass and relatively homogenous elites, for example.

5 Rawlsian Distribution

A lexmax (lexicographic maximum) model can be used to represent a wealth distribution that satisfies the Rawlsian difference principle. As before we let $u_i(x_i)$ be the social utility generated by a person i who initially receives wealth x_i. We also suppose that the fraction of total utility received by person i is proportion to the personal utility $v(x_i)$ of person i's initial wealth allocation. Thus everyone has the same personal utility function, even though different people may have different productivity functions.

If y_i is the utility enjoyed by person i, any solution of the following problem is a Rawlsian distribution:

$$\begin{array}{ll} \text{lexmax } y & (a) \\ \dfrac{y_i}{y_1} = \dfrac{v(x_i)}{v(x_1)},\ i=2,\ldots,n & (b) \\ \sum_{i=1}^n y_i = \sum_{i=1}^n u_i(x_i) & (c) \\ \sum_{i=1}^n x_i = 1 & (d) \\ x_i \geq 0,\ i=1,\ldots,n & (e) \end{array} \quad (18)$$

where $y = (y_1, \ldots, y_n)$. By definition, y^* solves (18) if and only if y_i^* solves problems $L_1, \ldots, L_n$, where L_k is the problem

$$\begin{array}{l} \max\ \min\{y_k, \ldots, y_n\} \\ (y_1, \ldots, y_{k-1}) = (y_1^*, \ldots, y_{k-1}^*) \\ (18b)\text{–}(18e) \end{array} \quad (19)$$

The lexmax solution is frequently defined with respect to a particular ordering $y_1, \ldots, y_n$ of the variables, in which case L_1 maximizes y_k rather than $\min\{y_k, \ldots, y_n\}$. This is inappropriate for the Rawlsian problem because we do not know in advance how the solution values y_k^* will rank in size.

Suppose, however, that persons $1, \ldots, n$ are indexed by increasing marginal productivity as in (3). Then we can assume without loss of generality that persons with less marginal productivity are nearer the bottom of the distribution.

Lemma 4 *Suppose that (3) holds and that $v(\alpha)$ is monotone nondecreasing for $\alpha \geq 0$. Then if (18) has a solution, it has a solution in which $y_1 \leq \cdots \leq y_n$.*

Proof. Since v is monotone, it suffices to show that (18) has a solution $(\bar{x}, \bar{y})$ in which $\bar{x}_1 \leq \cdots \leq \bar{x}_n$. For this it suffices to exhibit a solution $(\bar{x}, \bar{y})$ that solves L_k for $k = 1, \ldots, n$ and for which $\bar{x}_1 \leq \cdots \leq \bar{x}_n$.

Let (x^*, y^*) be a solution of (18), and let $(x^0, y^0) = (x^*, y^*)$. If $x_1^0 \leq x_i^0$ for $i = 2, \ldots, n$, then x^0 solves L_1 and we let $x^1 = x^0$. Otherwise we suppose $x_k^0 = \min_i\{x_i^0\}$ and define x^1 by $x_1^1 = x_k^0$, $x_k^1 = x_1^0$, and $x_i^1 = x_i^0$ for $i \neq 1, k$. We define y^1 to satisfy (18b)-(18c). We can see as follows that (x^1, y^1) solves L_1. If $U_0 = \sum_i u_i(x_i)$ is the total utility for solution (x^0, y^0), then the total utility for solution (x^1, y^1) is

$$U_1 = U_0 + u_k(x_1^0) - u_k(x_k^0) + u_1(x_k^0) - u_1(x_1^0)$$

But we have from (3) that

$$u_k(x_1^0) - u_1(x_k^0) \geq u_1(x_1^0) - u_1(x_k^0)$$

Thus $U_1 \geq U_0$, and x^1 generates no less total utility than x^0. Since utility is allotted to the y_i^1s in proportion to $v(x_i^1)$, and v is monotone nonincreasing, we get $y_1^1 \leq y_1^0$. Thus (x^1, y^1) solves L_1.

Now if $x_1^1 \leq x_i^1$ for $i = 2, \ldots, n$, then (x^1, y^1) solves L_1, L_2 and we let $(x^2, y^2) = (x^1, y^1)$. Otherwise we suppose $x_k^1 = \min_{i\geq 2}\{x_i^1\}$ and define x^2 by $x_1^2 = x_k^1$, $x_k^2 = x_1^1$, and $x_i^2 = x_i^1$ for $i > 2$ and $i \neq k$. We can show as above that (x^2, y^2) solves L_1, L_2. In this fashion we construct the sequence $(x^1, y^1), \ldots, (x^n, y^n)$ and let $(\bar{x}, \bar{y}) = (x^n, y^n)$. By construction, $\bar{x}_1 \leq \cdots \leq \bar{x}_n$. Since $(\bar{x}, \bar{y})$ solves $L_1, \ldots, L_n$, it solves (18).

To analyze solutions of (18), it is convenient to eliminate the variables y_i from each L_k. Using constraints (18b)–(18c), we get

$$y_i = v(x_i)\frac{\sum_{i=1}^n u_i(x_i)}{\sum_{i=1}^n v(x_i)},\ i = 1, \ldots, n$$

Using Lemma 4, L_k can be written

$$\begin{array}{ll} \max\ v(x_k)\dfrac{\sum_{i=1}^n u_i(x_i)}{\sum_{i=1}^n v(x_i)} & (a) \\ (x_1, \ldots, x_{k-1}) = (x_1^*, \ldots, x_{k-1}^*) & (b) \\ \sum_{i=1}^n x_i = 1 & (c) \\ x_k \leq \cdots \leq x_n & (d) \\ x_k \geq 0 & (e) \end{array} \quad (20)$$

where $x_1^*, \ldots, x_{k-1}^*$ are previously computed solutions of $L_1, \ldots, L_{k-1}$, respectively.

We focus first on L_1. Associating Lagrange multipliers $\mu_1, \ldots, \mu_{n-1}$ with the constraints in (20d), the KKT optimality conditions imply that a solution x with each $x_i > 0$ is optimal in (20) only if there are nonnegative values of $\mu_1, \ldots, \mu_{n-1}$ such that

$$\begin{array}{l} v'(x_1)\dfrac{\Sigma u}{\Sigma v} + v(x_1)\dfrac{u_1'(x_1)\Sigma v - v'(x_1)\Sigma u}{(\Sigma v)^2} - \lambda - \mu_1 = 0 \\ v(x_1)\dfrac{u_i'(x_i)\Sigma v - v'(x_i)\Sigma u}{(\Sigma v)^2} - \lambda + \mu_{i-1} - \mu_i = 0, \\ \qquad\qquad\qquad\qquad\qquad i = 2, \ldots, n-1 \\ v(x_1)\dfrac{u_n'(x_n)\Sigma v - v'(x_n)\Sigma u}{(\Sigma v)^2} - \lambda + \mu_{n-1} = 0 \end{array} \quad (21)$$

where

$$\Sigma u = \sum_{i=1}^n c_i u_i(x_i), \quad \Sigma v = \sum_{i=1}^n v(x_i)$$

and where $\mu_i = 0$ if $x_i < x_{i+1}$ in the solution.

We begin by examining the case in which each individual receives a different allotment x_i. Here each $\mu_i = 0$, and (21) implies

$$\frac{v'(x_1)}{v(x_1)} + \frac{u_1'(x_1)}{\Sigma u} - \frac{v_1'(x_1)}{\Sigma v} = \frac{u_i'(x_i)}{\Sigma u} - \frac{v_i'(x_i)}{\Sigma v}$$

for $i = 1, \ldots, n-1$, assuming $v(x_1) > 0$. This says that the marginal difference between productivity and personal utility is the same for everyone except the lowest ranked individual, for whom the difference is somewhat less. This tends to increase the allotment to the lowest individual, reducing the gap between this person and the others. The optimality conditions for L_2 are similar and likewise move the second closest individual closer to those who are more highly ranked. Thus in general, the lexmax solution results in a distribution that is more egalitarian than one in which the marginal difference between productivity and personal utility is the same for everyone.

6 Equality in a Rawlsian Distribution

We now examine conditions under which a Rawlsian distribution can be egalitarian. We found earlier that a utilitarian distribution with utility functions $u_i(x_i) = c_i x_i^p$, $v(x_i) = x_i^q$ cannot be egalitarian unless individuals are identical in their productivity. We will show that a Rawlsian distribution can, under certain conditions, be egalitarian in a more diverse population.

In an egalitarian distribution any μ_i can be nonzero. We eliminate λ from the optimality conditions (21) for L_1 to obtain

$$\begin{aligned} &\frac{v'(x_1)}{v(x_1)} + \frac{u_1'(x_1)}{\Sigma u} - \frac{v'(x_1)}{\Sigma v} - \frac{1}{v(x_1)}\frac{\Sigma v}{\Sigma u}\mu_1 \\ &= \frac{u_i'(x_i)}{\Sigma u} - \frac{v'(x_i)}{\Sigma v} + \frac{1}{v(x_1)}\frac{\Sigma v}{\Sigma u}(\mu_{i-1} - \mu_i) \end{aligned} \tag{22}$$

‘ for $i = 2, \ldots, n-1$, and

$$\begin{aligned} &\frac{v'(x_1)}{v(x_1)} + \frac{u_1'(x_1)}{\Sigma u} - \frac{v'(x_1)}{\Sigma v} - \frac{1}{v(x_1)}\frac{\Sigma v}{\Sigma u}\mu_1 \\ &= \frac{u_n'(x_n)}{\Sigma u} - \frac{v'(x_n)}{\Sigma v} + \frac{1}{v(x_1)}\frac{\Sigma v}{\Sigma u}\mu_{n-1} \end{aligned} \tag{23}$$

This yields the following.

Theorem 5 *Suppose the productivity functions are given by $u_i(\alpha) = c_i\alpha^p$ and the utility function by $v(\alpha) = \alpha^q$. Then L_1 has an egalitarian solution ($x_1 = \cdots = x_n$) only if*

$$\frac{1}{k}\sum_{i=1}^{k} c_i \geq \frac{1}{n-k}\sum_{i=k+1}^{n} c_i - \frac{q}{p}\cdot\frac{n-k}{k}\sum_{i=1}^{n} c_i \tag{24}$$

or equivalently,

$$\frac{1}{k}\sum_{i=1}^{k}\left(1 + \frac{q}{p}\cdot\frac{n-k}{k}\right)c_i \geq \frac{1}{n-k}\sum_{i=k+1}^{n}\left(1 - \frac{q}{p}\cdot\frac{n-k}{k}\right)c_i \tag{25}$$

for $k = 1, \ldots, n-1$.

Proof. The equations (22)–(23) can be written as (13) where

$$\begin{aligned} d_i = v(x_1)\frac{\Sigma u}{\Sigma v}\Bigg(&\frac{v'(x_1)}{v(x_1)} - \frac{u_{i+1}'(x_{i+1}) - u_1'(x_1)}{\Sigma u} \\ &+ \frac{v'(x_{i+1}) - v'(x_1)}{\Sigma v}\Bigg) \end{aligned}$$

for $i = 1, \ldots, n-1$. Substituting $x_1 = \cdots = x_n = 1/n$ and the functions u_i, v as given above, we obtain

$$d_i = qn^{-p}\sum_{j=1}^{n} c_j - pn^{-p}(c_{i+1} - c_1) \tag{26}$$

Since (15) solves (13), we can substitute (26) into (15) and get

$$\mu_k = p\frac{k(n-k)}{n^{1+p}}\left(\frac{q}{pk}\sum_{i=1}^{n} c_i + \frac{1}{k}\sum_{i=1}^{k} c_i - \frac{1}{n-k}\sum_{i=k+1}^{n} c_i\right)$$

for $k = 1, \ldots, n-1$. The KKT conditions imply that $x_k = \cdots = x_n = 1/n$ can be an optimal solution only if $\mu_k \geq 0$ for $k = 1, \ldots, n-1$, which implies (24).

An egalitarian solution ($x_1 = \cdots = x_n$) solves L_1 if and only if it solves the lexmax problem (18). If it solves L_1, then a lexmax solution must have $x_1 = 1/n$, which implies by (18d) that $x_2 = \cdots = x_n = 1/n$. If an egalitarian solution does not solve L_1, then some distribution with $x_1 < 1/n$ solves L_1, which implies that $x_1 < 1/n$ in any lexmax solution. Thus we have

Corollary 6 *If the productivity functions are given by $u_i(\alpha) = c_i\alpha^p$ and the utility function by $v(\alpha) = \alpha^q$, then a lexmax distribution is egalitarian ($x_1 = \cdots = x_n$) only if (24) and (25) hold.*

Thus a Rawlsian distribution is completely egalitarian when the gap between the average productivity of the k least productive individuals and that of the remaining population is not too great for any k. The maximum gap is proportional to q/p and $(n-k)/k$. This means that a smaller gap is required when the marginal utility of wealth decreases rapidly with the level of wealth (q is small), and when the opposite is true of marginal productivity (p is large). Thus an inegalitarian distribution is more likely when individuals do not care very much about getting rich and are satisfied with a moderate level of prosperity. Inequality is also more likely when allocating greater advantages to talented or industrious people reaps consistently greater rewards.

An egalitarian distribution also requires a smaller productivity gap between the highest class and the remaining population (i.e., when $k = n-1$) than between the lowest class and the remaining population ($k = 1$). Thus if the distribution of talents and industry has a long tail at the upper end, as is commonly supposed, the condition for equality could be hard to meet.

7 Conclusion

We find that a utilitarian distribution of wealth can result in substantial inequality when some individuals are more productive than others. The distribution is completely egalitarian only when every individual has the same marginal productivity. When marginal productivities are unequal, the most egalitarian distribution that is possible is one in which individuals are allocated wealth in proportion to their marginal productivity, and this occurs only when there are rapidly decreasing marginal returns for greater allocations of wealth.

A more egalitarian distribution results when the utility function includes a penalty to account for social dysfunction that inequality may cause. In particular, if the penalty is proportional to the gap between the richest and poorest individuals, we can calculate a constant of proportionality that results in a completely egalitarian distribution. This constant tends to be larger when there is large gap in average productivity between two segments of society. That is, there a group of individuals that have a much smaller average marginal productivity than the remaining individuals, relative to the overal range of productivities. This may occur, for example, when elites and common people form fairly homogenous groups separated by a large gap in average productivity.

Finally, the Rawlsian difference principle can result in a completely egalitarian distribution when no two segments of society have a large gap in average productivity. Equality is more likely to occur when there are decreasing returns for placing greater investment in talented and industrious people. Somewhat surprisingly, equality is also more likely when people are nearly as concerned about getting rich as about living a minimally comfortable lifestyle. When people want riches more, a privileged class is less likely to be consistent with Rawlsian justice. Finally, equality is more likely when the most talented and industrious individuals are not much more productive than the average person, even though the least productive individuals may fall far below the mean.

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