Hybrid Programs: Symmetrically Combining Natively Discrete and Continuous Truth-values

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Dedicated to Victor Marek on his 65th birthday

Abstract

The iterates $\mathbf{T}_P^t(I)$ of the one-step consequence operator $\mathbf{T}_P$ of a finite or infinite propositional normal logic program P applied to Herbrand interpretation I constitute a function $t \mapsto \mathbf{T}_P^t(I)$ from natural numbers to Herbrand interpretations. Without loss of generality, altering clause

$$p \leftarrow q_1, \ldots q_m, \neg r_1, \ldots \neg r_n$$

of P to

$$\frac{dp}{dt} = p \oplus (q_1, \ldots q_m, \neg r_1, \ldots \neg r_n)$$

amounts to regarding P as a system of first-order differential equations, where the mapping $t \mapsto \mathbf{T}_P^t(I)$ is a projection of the flow of the system with initial condition $t_0 \mapsto I$. The aim of this shift in viewpoint is to seamlessly combine logic program clauses with conventional first-order ordinary differential equations involving e.g. real-valued functions of a real variable. This is rigorously enabled by differentiation of functions that are morphisms in the category CONV of *convergence spaces*. The form of differentiation we describe is a *conservative extension* of differentiation of functions between familiar spaces associated with ordinary analysis. In particular, we can integrate logic programs over continuous time. In that case, stable models semantics provides the natural means to have an ordinary normal program be equivalent to its differential version.

1 Introduction

In prior work () the authors conservatively extended differentiation of functions arising in ordinary analysis, to differentiation of functions in the Cartesian-closed category of convergence spaces, CONV. Our purpose here is to apply the differentiation notion in CONV to seamlessly combine normal logic programs with ordinary differential equations to obtain hybrid programs.

By the idea of *conservative extension* we mean that differentiation is not altered for functions between the familiar spaces of ordinary analysis such as Euclidean vector spaces over the real numbers and Hilbert spaces over the complex numbers. The chain rule for differentiation in CONV holds: if

$$f : X \longrightarrow Y \text{ and } g : Y \longrightarrow Z$$

and df is a differential (not *derivative*) of f at x_0 and and dg is a differential of g at $f(x_0)$, then $dg \circ df$ is a differential of $g \circ f$ at x_0. It follows, for example, that if $X = Y = \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, then $dg \circ df$ is a conventional differential of a function on $\mathbb{R}$ at real number x_0, even if Y is some bizarre discrete/continuous hybrid convergence space. The notion of differential for functions on $\mathbb{R}$ has not been altered.

Convergence spaces include all topological spaces, and all reflexive directed graphs. The latter have multiple representations as convergence spaces. The morphisms of the category CONV are continuous functions between convergence spaces. The CONV notion of continuity is itself a conservative extension of the topological notion of continuity: If $f : X \longrightarrow Y$ is continuous in the CONV sense, and X and Y are topological spaces, then $f : X \longrightarrow Y$ is continuous in the topological sense, and conversely. The topological notion of continuity remains unaltered. In the case of all of the representations of reflexive directed graphs that we consider, CONV-continuity conservatively extends graph-homomorphism.

A word about derivatives: The derivative of a function at a point is a differential. For example, the derivative of $\lambda x \,.\, x^2$ at 1 is $\lambda x \,.\, 2x$, the linear function with slope 2. The derivative of a function f on a subset of the function's domain is another

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function that maps each point x of the subset to the derivative of f at x. The point is that derivatives are differential-valued. In the case of $\mathbf{E}^1$, the real numbers with the standard Euclidean topology, the space of linear functions, i.e. the space of differentials, is taken with a topology making it homeomorphic to $\mathbf{E}^1$. For situations where no such homeomorphism is available, we expect the codomain of a derivative of f to be different from the codomain of f. This is evident already with 2-dimensional vector spaces over the reals in ordinary analysis.

Plan of the paper: In the next section we discuss prior related work. In section 3 we introduce the syntax of hybrid programs. In section 4 we draw upon our earlier work in order to present enough about differentiation in CONV to provide a rigorous foundation for hybrid programs. In section 5 we discuss stable models for syntactically ordinary normal logic programs but integrated over continuous time.

2 Prior related work

There is a beautiful paper including a brief but powerful tutorial on convergence spaces due to R. Heckmann (Heckmann 2003). We highly recommend this paper.

The importance of the Cartesian-closed category of *convergence spaces* and continuous maps was introduced in (Kent 1964). Over time, a number of researchers have sought to generalize differentiability to spaces where the generalization is non-obvious. Some of the more serious and sophisticated results in this direction have employed one or another restriction of the notion of convergence space, often near to pre-topological spaces, or else stayed within TOP (Arens 1946; Averbukh & Smolyanov 1968; Binz 1966; Binz & Keller 1966; Fox 1945; Frölicher & Bucher 1966; Keller 1974; Kriegl 1983; Marinescu 1963; Michal 1938). These explorations assumed the existence of additional structure characterizing linearity. (Kriegl 1983) recognized the importance of Cartesian-closure for obtaining a robust chain-rule.

(Choquet 1947) studied what are now known as pretopological spaces. It is evident that every topological space is a pretopological space (cf. (Choquet 1947; Bourbaki textbf858 1940 textbf916 1942 textbf1029 1947 textbf1045 1948 textbf1084 1949; Kelley 1955)). and that the convergence space notion of continuity is a conservative extension of the topological space notion of continuity. (Bordaud 1979) proved that a certain 3-point pretopological space is universal for all pretopological spaces.

The convergence structure of function spaces that provides for the Cartesian closure of CONV is traceable to (Katětov 1965).

Boolean derivatives (Reed 1954; Akers Jr. 1959; Vichniac 1990) are a specific special case of differentiation in the category CONV.

The earliest traces in the direction of convergence spaces that we are aware of are due to Hausdorff, (Hausdorff 1935) and Bertrand Russell. Hausdorff studied spaces he called *Gestufte Räume* - a term rife with allusions to art deco interiors of restaurants and night clubs of the era. This was preceded by an effort of Bertrand Russell (Russell 1919) in which he attempted to characterize continuity purely in terms of *intervals* within relations.

Bill Rounds and Hosung Song (Reidys & Stadler 2003) developed Hybrid Pi-Calculus, which informed the notion of *hybrid program* presented here.

For basic notions and fundamental results concerning logic programs and stable models the reader is referred to *Nonmonotonic Logic* by Wiktor Marek and Miroslaw Truszczynski (V. W. Marek 1993).

3 Hybrid programs

In this section we begin with a slight variant of ordinary normal ground logic programs that are equivalent to ordinary ground normal logic programs, but that avoid a potential inconsistency when programs are transformed to their differential versions. We then give the syntax of *differential* propositional normal logic programs, and show how differential normal logic programs are equivalent to normal logic programs. Next we add in first-order ordinary differential equations to obtain hybrid programs.

Let P be a finite non-ground normal logic program, with possibly infinitely many clauses. Each clause in P has the form

$$p \leftarrow \beta$$

where β is a formula $q_1, \ldots q_m, \neg r_1, \ldots \neg r_n$ where p, $q_1, \ldots q_m$, $r_1, \ldots r_n$ are ground atoms, after possibly reordering the literals of β. β is called the *body* of the clause, and p is its *head*. p occurs as the head of possibly infinitely many clauses of P. For each atom p that occurs as the head of some clause in P, we define, following (Clark 1978), the *definition of* p to be

$$p \leftarrow \bigvee\{\beta \mid p \leftarrow \beta \text{ is a clause in } P\}$$

where $\bigvee$ denotes possibly infinitary disjunction. We will overload the term *propositional normal logic program* to allow for programs whose clauses have the form given for the definition of p. Given a propositional normal logic program P, let def(P) be the program that consists of clauses that are definitions of atoms that occur as the head of some clause in P.

The iterates $\mathbf{T}_P^t(I)$ of the one-step consequence operator $\mathbf{T}_P$ (Lloyd 1987) of a finite or infinite propositional normal logic program P applied to Herbrand interpretation I constitute a function $t \mapsto \mathbf{T}_P^t(I)$ from natural numbers to Herbrand interpretations.

Proposition 3.1

$$\mathbf{T}_P^t(I) = \mathbf{T}_{\mathrm{def}(P)}^t(I)$$

∎

Now consider the expression

$$\frac{dp}{dt} = \gamma$$

where p is a ground atom and γ is a possibly infinitary disjunction of finite conjunctions of literals. We call the above expression a *differential clause*, and extend the usage of *head* and *body* in the obvious way. We call a collection of such clauses a *differential program*. A *solution* of a differential program is a mapping $t \mapsto I_t$, where t varies over the integers and each I_t is an Herbrand interpretation for the language of the program, such that at each t', γ in $I_{t'}$ is the differential of $I_{t'}(p)$, for each ground atom p in the language of the program. We regard an Herbrand interpretation as a mapping from the ground atoms of the program to $\{\mathsf{f}, \mathsf{t}\}$, the set of truth-values. For this to make sense, we must be able to identify the differentials of functions from the integers to $\{\mathsf{f}, \mathsf{t}\}$ with the members of $\{\mathsf{f}, \mathsf{t}\}$. We will return to this problem after the next section in which we define differentials.

Definition 3.1 *Let $\leq$ be the reflexive relation on $\{\mathsf{f}, \mathsf{t}\}$ such that let $\mathsf{f} \leq \mathsf{t}$. $\leq$ on $\{\mathsf{f}, \mathsf{t}\}$ lifts pointwise to an ordering $\leq$ on Herbrand interpretations.*

For the present, assume we can make this identification and that if $\frac{dp}{dt} \upharpoonright_{t=t'} = \mathsf{f}$, then $I_{t'+1}(p) = I_{t'}(p)$, and if $\frac{dp}{dt} \upharpoonright_{t=t'} = \mathsf{t}$, then $I_{t'+1}(p) \neq I_{t'}(p)$. Then

Theorem 3.1 *Let P be a propositional normal logic program in which each clause head occurs in only one clause of P. Suppose that the mapping $t \mapsto \mathbf{T}_P^t(I)$ is monotonic. Let Q result from P by replacing each clause $p \leftarrow \gamma$ of P with the differential program clause*

$$\frac{dp}{dt} = \gamma$$

Then,

$$t \mapsto \mathbf{T}_P^t(I)$$

is the least monotonic solution of Q such that $0 \mapsto I$. ∎

The restriction to least monotonic solutions in the previous proposition arises from the convergence structure on the space of the two differentials with which the members of $\{\mathsf{f}, \mathsf{t}\}$ are identified, cf. the next section.

Each clause body of a differential program can be regarded as a function from Herbrand interpretations to $\{\mathsf{f}, \mathsf{t}\}$. An Herbrand interpretation as a mapping from an Herbrand base to $\{\mathsf{f}, \mathsf{t}\}$ is an element of a Cartesian power of $\{\mathsf{f}, \mathsf{t}\}$ indexed by the Herbrand base, $\{\mathsf{f}, \mathsf{t}\}^H$. Thus each clause body is a function from a suitable Cartesian power of $\{\mathsf{f}, \mathsf{t}\}$ to $\{\mathsf{f}, \mathsf{t}\}$. For a hybrid program P we use two kinds of clause bodies: Choose a collection of real-valued variables Λ. The domain of each clause body is $\{\mathsf{f}, \mathsf{t}\}^H \times \mathbb{R}^\Lambda$. The first kind of clause body maps $\{\mathsf{f}, \mathsf{t}\}^H \times \mathbb{R}^\Lambda$ to $\{\mathsf{f}, \mathsf{t}\}$. The second kind maps $\{\mathsf{f}, \mathsf{t}\}^H \times \mathbb{R}^\Lambda$ to $\mathbb{R}$. In each case we will be identifying the return values of the clause bodies with differentials of suitable type, depending on whether t is integer or real-valued.

4 Differentiation

We begin with a brief description of the category CONV: A *filter* on a set X is a nonempty collection of subsets of X closed under finite intersection and reverse inclusion. $\mathcal{F}$ is a *proper* filter if the empty set is not a member of $\mathcal{F}$. Let $\Phi(X)$ denote the set of all filters on X. For a subset A of X, $\{ B \mid A \subseteq B \subseteq X \}$ is a member of $\Phi(X)$. We denote this filter by $[A]$. In the special case where A is a singleton $\{x\}$ we denote $[A]$ by $[x]$ and call this the *point filter* at x.

Definition 4.1 *(Kent 1964; Heckmann 2003) A* convergence structure *on X is a relation $\downarrow$ (read as "converges to") between members of $\Phi(X)$ and members of X such that for each $x \in X$: (1) $[x]$ converges to x, and (2) the set of filters converging to x is closed under reverse inclusion. A pair $(X, \downarrow)$ consisting of a set X and a convergence structure $\downarrow$ on X is called a* convergence space.

A function $f : X \longrightarrow Y$ where X and Y are sets, induces functions $\hat{f} : 2^X \longrightarrow 2^Y$ and $\hat{\hat{f}} : \Phi(X) \longrightarrow \Phi(Y)$. $\hat{f}$ is defined by $\hat{f}(A) = \{f(a) \mid a \in A\}$, which we call the f-*image* of A. For $\mathcal{F} \in \Phi(X)$ note that the collection of all supersets of f-images of members of $\mathcal{F}$ forms a filter which we call $\hat{\hat{f}}(\mathcal{F})$. Hereafter we overload notation and drop the $\hat{}$ and $\hat{\hat{}}$ annotations.

When convenient, we will refer to a convergence space $(X, \downarrow)$ by its carrier, X.

Definition 4.2 *(Kent 1964; Heckmann 2003) Let $f : X \longrightarrow Y$ where X and Y are convergence spaces, and let $x_0 \in X$. f is* continuous at x_0 *iff for each $\mathcal{F} \in \Phi(X)$, if $\mathcal{F} \downarrow x_0$ in X, then $f(\mathcal{F}) \downarrow f(x_0)$ in Y. f is* continuous *iff f is continuous at every point of X.*

Continuity can be characterized in terms of filter members, which play a role analogous to the role played by neighborhoods, as supersets of open sets, in topological spaces.

Proposition 4.1 *Let $f : X \longrightarrow Y$ where X and Y are convergence spaces, and let x_0 be a point of X. f is continuous at x_0 iff for every filter $\mathcal{F}$ converging to x_0 in X, there is a filter $\mathcal{G}$ converging to $f(x_0)$ in Y such that $(\forall V \in \mathcal{G})(\exists U \in \mathcal{F})[f(U) \subseteq V]$.*

Definition 4.3 *(Kent 1964) A* homeomorphism *between two convergence spaces is a continuous bijection whose inverse is continuous.*

The objects of the category of convergence spaces CONV are the convergence spaces. For convergence spaces X and Y, HOM(X, Y) is the set of continuous functions from X to Y.

Definition 4.4 *Let x be a point of a convergence space X, and let U be a subset of X. U is said to be a* neighborhood *of x iff U belongs to every filter converging to x.*

Definition 4.5 *(Choquet 1947) A convergence space $(X, \downarrow)$ is said to be a* pretopological space *if and only if $\downarrow$ is a* pretopology*, i.e. for each $x \in X$, the collection of all neighborhoods of x converges to x.*

Proposition 4.2 *Let $f : X \longrightarrow Y$ where X and Y are pretopological spaces, and let $x_0 \in X$. f is continuous at x_0 iff for every neighborhood V of $f(x_0)$, there is a neighborhood U of x_0 such that $f(U) \subseteq V$.*

There are multiple ways of representing a reflexive directed graph as a convergence space. Two of these representations are centrally important for our purposes. The first way is to have a pretopological space all of whose neighborhood filters are principal; that is, has a smallest member. Then each point in the space has a smallest neighborhood containing the point, which serves as the digraph neighborhood of the point. The second way requires a little setting up.

Definition 4.6 *A convergence space X will be said to be* postdiscrete *if and only if every convergent proper filter is a point filter.*

Definition 4.7 *Let (V, E) be a reflexive digraph. Induce a convergence structure on V by letting a proper filter $\mathcal{F}$ converge to a vertex x iff $\mathcal{F} = [y]$ for some vertex y with an edge in E from x to y.*

It is readily verified that if (V_1, E_1) and (V_2, E_2) are reflexive digraphs, then a function $f : V_1 \longrightarrow V_2$ is continuous (with respect to the induced convergence structures on V_1 and V_2) iff, for all edges (x, y) in E_1, the edge $(f(x), f(y))$ is present in E_2.

Proposition 4.3 *The reflexive digraphs whose induced pretopologies are topological are precisely those in which the underlying binary relation is transitive as well as reflexive.*

Unlike **TOP** and **PreTOP**, **CONV** is a Cartesian closed category ((MacLane 1971; Arbib & Manes 1975; Adámek, Herrlich, & Strecker 1990; Schröder 2001; Katětov 1965)):

Definition 4.8 *(Katětov 1965)*
Let X and Y be convergence spaces. The function space Y^X *is the set of all continuous functions from X to Y, equipped with the convergence structure $\downarrow$ defined as follows: For each $\mathcal{H} \in \Phi(Y^X)$ and each $f_0 \in Y^X$, let $\mathcal{H} \downarrow f_0$ if, and only if, for each $x_0 \in X$ and each $\mathcal{F} \downarrow x_0$, $\{\, \{\, f(x) \mid f \in H, x \in F \,\} \mid H \in \mathcal{H}, F \in \mathcal{F} \,\}$ is a base for a filter which converges to $f(x_0)$ in Y.*

Just as continuity itself neither presupposes any separation strength nor any notion of linearity, neither does differentiability. The familiar differential calculus on Euclidean spaces is of course intrinsically dependent on the vector space structure, but this is due to the choice of functions used to serve as differentials, and the consequent determination of the conditions under which functions are differentiable. What matters is the differentiability relation "differential g **is a differential of** f **at** x". Unless we demand of g that it satisfy some kind of linearity property, linearity does not intrinsically enter into the relation.

The central idea in setting up differentiation in CONV is to uniformly define the 3-place relation

___ is a differential of ___ at ___

for each pair of convergence spaces X, Y in the category, where the first and second arguments are elements of Hom(X, Y) and the third argument is an element of X, in such a way as to (1) obtain the chain rule and (2) have the relation be in agreement with standard definitions in real and complex analysis.

We first need to extract from the convergence structure of each space how to *translate* a point in the space.

Definition 4.9 *An* automorphism *of a convergence space X is a homeomorphism $f : X \longrightarrow X$.*

Definition 4.10 *A* translation group *on a convergence space X is a group T of automorphisms of X such that, for each pair of points p and q of X, there is at most one member of T which maps p to q.*

The action of a translation group on X is called a *semi-regular action* on X.

A translation group T on X partitions X into orbits. For $x \in X$, the orbit containing x is $\{t(x) \mid t \in T\}$. From each orbit select one member, called the *origin* of the orbit. The origin of the orbit containing x is denoted by 0_x. The member of T that maps 0_x to x is denoted by τ_x. For any two points x and y of X, let $x + y = \tau_y(x)$, and $-y = \tau_y^{-1}(0_y)$. If T is Abelian, and x are in the same orbit, then $x+y = y+x$. If T is not Abelian, then $x + y$ does not have to commute, and never commutes when x and y lie in distinct orbits. $x-y$ abbreviates $x + (-y)$.

The set of all origins of the orbits of T is called a transversal with respect to T. For each convergence space select a (possibly trivial) translation group and a transversal. Choose a subcategory of CONV by restricting every Hom(X, Y) to set $D(X, Y)$ so that (1) every member of $D(X, Y)$ preserves origins: If $f \in D(X, Y)$ then for every $x \in X$, $f(0_x)$ is an origin in Y.

Let $L \in \mathcal{D}(\mathcal{X}, \mathcal{Y})$, where $\mathcal{D}(\mathcal{X}, \mathcal{Y})$ is the set of all arrows in $\mathcal{D}$ from $\mathcal{X}$ to $\mathcal{Y}$, equipped with the subspace convergence structure inherited from the function space Y^X in **CONV**.

Definition 4.11 *L is a* differential *of f at a iff for every $\mathcal{F} \downarrow a$ in X, there is some $\mathcal{H} \downarrow L$ in $\mathcal{D}(\mathcal{X}, \mathcal{Y})$ such that*

i. *$\mathcal{H} \subseteq [\{L\}]$, and*

ii. *for every $H \in \mathcal{H}$, there is some $F \in \mathcal{F}$ such that for every point $x \in F$, there is at least one function $t \in H$ such that*

$$t(x - a) = f(x) - f(a)$$

Theorem 4.1 ***(Chain Rule)*** *Suppose that f is continuous at a. Also suppose that K is a differential of f at a, and L is a differential of g at $f(a)$. Then $L \circ K$ is a differential of $g \circ f$ at a.*

5 Differentials for hybrid programs and stable models semantics

In the section on hybrid programs we were left with the need for choosing two differentials of the continuous functions from the set of integers to the set $\{\mathtt{f}, \mathtt{t}\}$, and two differentials of the continuous functions from the set of real numbers to the set $\{\mathtt{f}, \mathtt{t}\}$. We equip the integers with the reflexive digraph structure arising from the successor relation, the set of real numbers with the standard Euclidean topology, and the set $\{\mathtt{f}, \mathtt{t}\}$ with the Sierpinski topology, with $\{\mathtt{t}\}$ open. In both cases we choose the constant function corresponding to $\mathtt{f}$ and the constant function corresponding to $\mathtt{t}$. The resulting 2-point set will be equipped with the Sierpinski topology with the singleton of the element corresponding to 1 as the open singleton. The effect of this convergence structure on the set of differentials where t varies over the integers is that whenever the differential $\frac{dp}{dt}$ is required to be the constant function corresponding to $\mathtt{t}$, the value of p is $\mathtt{t}$ and is required to remain so for at least one time-step. But, whenever the value of $\frac{dp}{dt}$ is $\mathtt{t}$, p is unconstrained at that moment. In the case where time t varies over $\mathbb{R}$, whenever the value of $\frac{dp}{dt}$ is $\mathtt{t}$, the value of p must be $\mathtt{t}$ over some (think small) time interval containing t_0. There are many solutions of the system, nondeterministically. But if we restrict the admissible solutions to be monotonic, the concluding theorem of the section on hybrid programs now follows.

In the theorem, if one did not begin with a program that produced a monotonic trajectory when started on the least Herbrand interpretation $\perp$, one may then seek Gelfond-Lifschitz transforms corresponding to stable model guesses of the program which do produce monotonic trajectories when started on $\perp$. Each differential program corresponding to such a Gelfond-Lifschitz transform then produces a least trajectory (i.e. slowest growing) whose least upper bound will be the original guess iff the original guess was stable.

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